

# RDWT and Image Watermarking

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## 1. Introduction

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Image watermarking is a technique for labeling digital images by embedding electronic stamps or so-called watermarks into the images for the purpose of copyright protection. Due to the explosion in use of the digital media, watermarking has been attracting significant interest from both academic and industry recently. In order to be effective, a watermark should exhibit a number of desirable characteristics [1, 2, 3, 4].

- *Unobtrusiveness*: The watermark should be perceptually invisible and should not degrade the quality of the image content.
- *Robustness*: The watermark must be robust to transformations including common signal processings, common geometric distortions and subterfuge attacks.
- *Blindness*: The watermark should not require the original nonwatermarked image for watermark detection.
- *Unambiguousness*: Retrieval of the embedded watermark should unambiguously identify the ownership and distribution of data.

A number of techniques have been developed for watermarking. A widely used technique is spread-spectrum watermarking which embed white Gaussian noise onto transform coefficients [1]. The watermark is detected by computing a correlation between the watermarked coefficients and the watermark sequence which is compared to a properly selected threshold. The DWT is an appealing transform for spread-spectrum watermarking because its space-frequency tiling exhibits a strong similarity to the way the human visual system (HVS) processes natural images [4]. Therefore, watermarking techniques in the wavelet domain can largely exploit the HVS characteristics and effectively hide a robust watermark.

Unlike the DWT, the redundant discrete wavelet transform (RDWT) gives an overcomplete representation of the input sequence and functions as a better approximation to the continuous wavelet transform. The RDWT is shift invariant and its redundancy introduces an overcomplete frame expansion. It has been proved that frame expansions add numerical robustness in case of adding white noise [5, 6] such in the case of quantization. This property makes RDWT-based signal processing tends to be more robust than DWT. It is well known that RDWT is very successful in noise reduction and feature detection, and prior work has proposed using the RDWT for image watermarking [7]. Initially, one might think that, since frame expansions such as the RDWT offer increased robustness to added noise, such overcomplete expansions outperform traditional orthonormal expansion in the watermarking problem. In this report, we offer analysis that contradicts this intuition. Specifically, we present analysis that shows that, although watermarking coefficients in a tight-frame expansion does produce less image distortion for the same watermarking energy, the correlation-detector performance of tight-frame based watermarking is identical to that obtained by using an orthonormal expansion.

## 2. RDWT & Frame

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### 2.1 RDWT

The RDWT removes the decimation operators from DWT filter banks. To retain the multiresolution characteristic, the wavelet filters must be adjusted accordingly at each scale. Specifically,

$$h_{J_1}[k] = h[k]$$

where  $J_1$  is the start scale,  $h_{J_1}[k]$  is the RDWT scaling filter at scale  $J_1$ .  $h[k]$  is a normal DWT scaling filter.

Filters at later scales are upsampled versions from the filter coefficients at the upper stage,

$$h_j[k] = h_{j+1}[k] \uparrow 2$$

and similar definitions is applied to  $g_j[k]$ , the wavelet filter of the orthonormal DWT.

The RDWT multiresolution analysis can be implemented via the filter bank equations:

$$\text{Analysis:} \quad c_j[k] = \tilde{h}_{j+1}[-k] * c_{j+1}[k] \quad (1)$$

$$d_j[k] = \tilde{g}_{j+1}[-k] * d_{j+1}[k] \quad (2)$$

$$\text{Synthesis:} \quad c_{j+1}[k] = \frac{1}{2} [h_{j+1}[k] * c_j[k] + g_{j+1}[k] * d_j[k]] \quad (3)$$

The lack of downsampling in the RDWT analysis yields a redundant representation of the input sequence; specifically, two valid descriptions of the coefficients exist after one stage of RDWT analysis.

### 2.2 Frames

DEFINITION. [5] A family of functions  $(\psi_i)_{i \in J}$  in a Hilbert space  $\mathcal{H}$  is called a frame if there exist  $A > 0$ , and  $B < \infty$  so that, for all  $f$  in  $\mathcal{H}$ ,

$$A\|f\|^2 \leq \sum_{i \in J} |\langle \psi_i, f \rangle|^2 \leq B\|f\|^2 \quad (4)$$

$A$  and  $B$  are called the *frame bounds*. The *dual frame*  $(\tilde{\psi}_i)$  of  $(\psi_i)$  is an expansion set in Hilbert space  $\mathcal{H}$  and for all  $f$  in  $\mathcal{H}$ ,

$$\frac{1}{B}\|f\|^2 \leq \sum_i |\langle \tilde{\psi}_i, f \rangle|^2 \leq \frac{1}{A}\|f\|^2 \quad (5)$$

Any function  $f \in \mathcal{H}$  can be expanded as

$$f = \sum_i \alpha_i \tilde{\psi}_i = \sum_i \langle \psi_i, f \rangle \tilde{\psi}_i \quad (6)$$

$$= \sum_i \beta_i \psi_i = \sum_i \langle \tilde{\psi}_i, f \rangle \psi_i \quad (7)$$

If two frame bounds are equal,  $A = B$ , the frame is called a *tight frame*. In a tight frame, for all  $f \in \mathcal{H}$ ,

$$\sum_{i \in J} |\langle \psi_i, f \rangle|^2 = A\|f\|^2 \quad (8)$$

$$\tilde{\psi}_i = \frac{1}{A} \psi_i \quad (9)$$

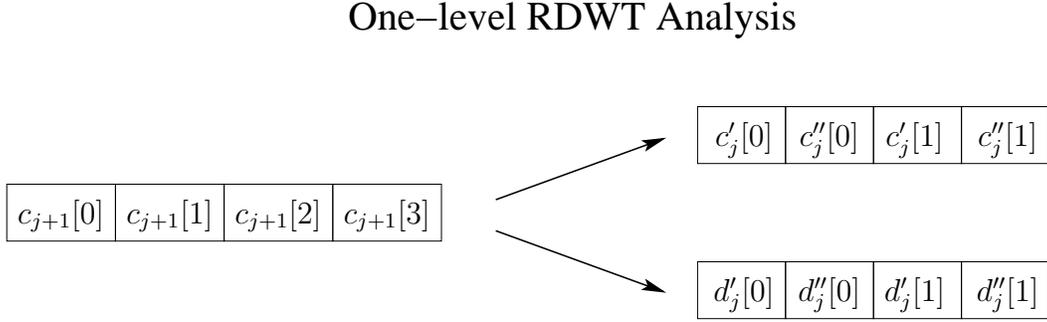
$$f = \frac{1}{A} \sum_i \langle \psi_i, f \rangle \psi_i \quad (10)$$

In this case,  $A > 1$ , and  $A$  gives the “redundancy ratio”, a measure of the degree of overcompleteness of the expansion.

### 2.3 RDWT and frame expansion

*Theorem.* RDWT is a frame expansion with frame bounds  $A = 2$  and  $B = 2^J$ , where  $J$  is the number of levels in the transform. Thus, for one level, the RDWT is a tight frame.

*Proof:*



**Figure 1:** One scale of RDWT decomposition.

As shown in Fig. 1, for the lowpass coefficients  $c_j$ , it is composed with two parts,  $c'_j$  and  $c''_j$ , each of them is a valid DWT lowpass description of  $c_{j+1}$ . It is a similar case for the highpass coefficients  $d_j$ . Thus, Parseval's theory holds for each of the descriptions:

$$\begin{aligned}\sum \|c'_j\|^2 + \sum \|d'_j\|^2 &= \sum \|c_{j+1}\|^2 \\ \sum \|c''_j\|^2 + \sum \|d''_j\|^2 &= \sum \|c_{j+1}\|^2\end{aligned}$$

Thus, for the RDWT coefficients all together, we have

$$\sum \|c_j\|^2 + \sum \|d_j\|^2 = 2 \sum \|c_{j+1}\|^2 \quad (11)$$

Therefore, one-level RDWT decomposition is a tight frame with  $A = 2$ .

For decomposition level  $J > 1$ , suppose the decomposition starts at scale  $J_1$ . We have then

$$\begin{aligned}\sum \|c_{J_1}\|^2 &= \frac{1}{2}(\sum \|c_{J_1-1}\|^2 + \sum \|d_{J_1-1}\|^2) \\ &= \frac{1}{2^2} \sum \|c_{J_1-2}\|^2 + \frac{1}{2^2} \sum \|d_{J_1-2}\|^2 + \frac{1}{2} \sum \|d_{J_1-1}\|^2 \\ &= \frac{1}{2^J} \sum \|c_{J_1-J}\|^2 + \sum_{j=1}^J \frac{1}{2^j} \sum \|d_{J_1-j}\|^2\end{aligned} \quad (12)$$

While, the energy for the RDWT coefficients is:

$$E = \sum \|c_{J_1-J}\|^2 + \sum_{j=1}^J \sum \|d_{J_1-j}\|^2 \quad (13)$$

So that,

$$\begin{aligned}
2^J \sum \|c_{J_1}\|^2 - E &= \sum_{j=1}^{J-1} (2^{J-j} - 1) \sum \|d_{J_1-j}\|^2 \\
&\text{(since } j = 1 \dots J-1, \text{ we have } 2^{J-j} \geq 2, \text{ so that)} \\
2^J \sum \|c_{J_1}\|^2 - E &\geq 0 \\
\implies E &\leq 2^J \sum \|c_{J_1}\|^2
\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}
E - 2 \sum \|c_{J_1}\|^2 &= (1 - 2^{1-J}) \sum \|c_{J_1-J}\|^2 + \sum_{j=2}^J (1 - 2^{1-j}) \sum \|d_{J_1-j}\|^2 \\
&\text{(since } J > 1 \text{ and } j = 2 \dots J, \text{ we have } 2^{1-j} \leq 1, \text{ so that)} \\
E - 2 \sum \|c_{J_1}\|^2 &\geq 0 \\
\implies E &\geq 2 \sum \|c_{J_1}\|^2
\end{aligned} \tag{15}$$

The bounds of  $A = 2$  and  $B = 2^J$  are the tightest bounds since we can find sequences that meet the bounds. Specifically, for a constant sequence  $x_1[n] = 1$ , only the lowpass coefficients are nonzero and all highpass subbands are zero valued. That is,  $\sum \|c_{J_1-j}\|^2 \neq 0$  but  $\sum \|d_{J_1-j}\|^2 = 0$ , for  $j = 1 \dots J$ .

$$\begin{aligned}
2^J \sum \|c_{J_1}\|^2 - E &= \sum_{j=1}^{J-1} (2^{J-j} - 1) \sum \|d_{J_1-j}\|^2 = 0 \\
\therefore E &= 2^J \sum \|c_{J_1}\|^2
\end{aligned} \tag{16}$$

For an oscillatory sequence  $x_2[n] = (-1)^n$ , only the finest detail coefficients would be nonzero. That is,  $\sum \|c_{J_1-j}\|^2 = 0$  and  $\sum \|d_{J_1-j}\|^2 = 0$ , for  $j = 2 \dots J$ .

$$\begin{aligned}
E - 2 \sum \|c_{J_1}\|^2 &= (1 - 2^{1-J}) \sum \|c_{J_1-J}\|^2 + \sum_{j=2}^J (1 - 2^{1-j}) \sum \|d_{J_1-j}\|^2 = 0 \\
\therefore E &= 2 \sum \|c_{J_1}\|^2
\end{aligned} \tag{17}$$

Therefore, for decomposition level  $J > 1$ , the energy of the decomposition coefficients are well bounded. Thus, the RDWT is a frame expansion according to the definition. ■

### 3. Robustness of Adding White Noise

We are to compare the robustness of three different transforms, i.e. orthonormal basis, tight frame and frame basis, after adding white Gaussian noise in the corresponding transform domains. Suppose the additive noise is a zero mean, variance  $\epsilon^2$  Gaussian noise, the robustness is measured as the mean square error (MSE) of the reconstructed signal with the original signal.

$$\begin{aligned}
MSE &= E[\|f - \hat{f}\|^2] = E[\langle f - \hat{f}, f - \hat{f} \rangle] \\
&= E[\langle f, f \rangle] - 2E[\langle f, \hat{f} \rangle] + E[\langle \hat{f}, \hat{f} \rangle]
\end{aligned}$$

In this analysis,  $f$  is the original signal,  $\hat{f}$  is the watermarked signal, and the MSE is the distortion produced by watermarking  $f$  with watermark energy equal to  $\epsilon^2$ .

### 3.1 Orthonormal basis

$$f = \sum_{i=1}^N \alpha_i \psi_i = \sum_{i=1}^N \langle \psi_i, f \rangle \psi_i$$

$$\hat{f} = \sum_{i=1}^N (\alpha_i + n_i) \psi_i$$

where the Gaussian noise is  $n_i \sim (0, \epsilon^2)$ .

The error signal

$$e = \hat{f} - f = \sum_{i=1}^N n_i \psi_i$$

So that,

$$MSE = E\|e\|^2 = E\left[\sum_{i=1}^N n_i^2\right]$$

$$= \sum_{i=1}^N E[n_i^2] = N\epsilon^2 \quad (18)$$

### 3.2 Tight frame

$$f = \frac{1}{A} \sum_{i=1}^{AN} \langle \psi_i, f \rangle \psi_i = \frac{1}{A} \sum_{i=1}^{AN} \alpha_i \psi_i$$

$$\hat{f} = \frac{1}{A} \sum_{i=1}^{AN} (\alpha_i + n_i) \psi_i$$

Since

$$MSE = E[\langle f, f \rangle] - 2E[\langle f, \hat{f} \rangle] + E[\langle \hat{f}, \hat{f} \rangle]$$

(1)

$$E[\langle f, f \rangle] = E\left[\left\langle \frac{1}{A} \sum_{i=1}^{AN} \alpha_i \psi_i, \frac{1}{A} \sum_{j=1}^{AN} \alpha_j \psi_j \right\rangle\right]$$

$$= \frac{1}{A^2} \sum_i \sum_j E[\alpha_i \alpha_j] \langle \psi_i, \psi_j \rangle$$

(2)

$$E[\langle f, \hat{f} \rangle] = E\left[\left\langle \frac{1}{A} \sum_{i=1}^{AN} \alpha_i \psi_i, \frac{1}{A} \sum_{j=1}^{AN} (\alpha_j + n_j) \psi_j \right\rangle\right]$$

$$= \frac{1}{A^2} \sum_i \sum_j E[\alpha_i \alpha_j] \langle \psi_i, \psi_j \rangle + \frac{1}{A^2} \sum_i \sum_j E[\alpha_i n_j] \langle \psi_i, \psi_j \rangle$$

$$= \frac{1}{A^2} \sum_i \sum_j E[\alpha_i \alpha_j] \langle \psi_i, \psi_j \rangle$$

(3)

$$\begin{aligned}
E[\langle \hat{f}, \hat{f} \rangle] &= E \left[ \left\langle \frac{1}{A} \sum_{i=1}^{AN} (\alpha_i + n_i) \psi_i, \frac{1}{A} \sum_{j=1}^{AN} (\alpha_j + n_j) \psi_j \right\rangle \right] \\
&= \frac{1}{A^2} \sum_i \sum_j E[\alpha_i \alpha_j] \langle \psi_i, \psi_j \rangle + 0 + \frac{1}{A^2} \sum_i \sum_j E[n_i n_j] \langle \psi_i, \psi_j \rangle
\end{aligned}$$

Combining the three components:

$$\begin{aligned}
MSE &= E[\|f - \hat{f}\|^2] = E[\langle f, f \rangle] - 2E[\langle f, \hat{f} \rangle] + E[\langle \hat{f}, \hat{f} \rangle] \\
&= \frac{1}{A^2} \sum_i \sum_j E[n_i n_j] \langle \psi_i, \psi_j \rangle \\
&= \frac{1}{A^2} \sum_i E[n_i^2] \langle \psi_i, \psi_i \rangle \\
&= \frac{1}{A^2} \sum_{i=1}^{AN} \epsilon^2 = \frac{N\epsilon^2}{A}
\end{aligned} \tag{19}$$

Note that, since  $A > 1$ ,  $\frac{N\epsilon^2}{A} < N\epsilon^2$ , and watermarking in the tight frame yields less distortion to  $f$  than does an orthonormal basis.

### 3.3 Frame expansion

$$\begin{aligned}
f &= \sum_{i=1}^M \alpha_i \tilde{\psi}_i = \sum_{i=1}^M \langle \psi_i, f \rangle \tilde{\psi}_i \\
\hat{f} &= \sum_{i=1}^M (\alpha_i + n_i) \tilde{\psi}_i
\end{aligned}$$

$$MSE = E[\langle f, f \rangle] - 2E[\langle f, \hat{f} \rangle] + E[\langle \hat{f}, \hat{f} \rangle]$$

(1)

$$\begin{aligned}
E[\langle f, f \rangle] &= E[\langle \sum_i \alpha_i \tilde{\psi}_i, \sum_j \alpha_j \tilde{\psi}_j \rangle] \\
&= \sum_i \sum_j E[\alpha_i \alpha_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle
\end{aligned}$$

(2)

$$\begin{aligned}
E[\langle f, \hat{f} \rangle] &= E[\langle \sum_i \alpha_i \tilde{\psi}_i, \sum_j (\alpha_j + n_j) \tilde{\psi}_j \rangle] \\
&= \sum_i \sum_j E[\alpha_i \alpha_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle + \sum_i \sum_j E[\alpha_i n_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle \\
&= \sum_i \sum_j E[\alpha_i \alpha_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle
\end{aligned}$$

(3)

$$\begin{aligned}
E[\langle \hat{f}, \hat{f} \rangle] &= E[\langle \sum_i (\alpha_i + n_i) \tilde{\psi}_i, \sum_j (\alpha_j + n_j) \tilde{\psi}_j \rangle] \\
&= \sum_i \sum_j E[\alpha_i \alpha_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle + \sum_i \sum_j E[n_i n_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle \\
&= \sum_i \sum_j E[\alpha_i \alpha_j] \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle + \sum_i E[n_i^2] \langle \tilde{\psi}_i, \tilde{\psi}_i \rangle
\end{aligned}$$

Combining all the components:

$$\begin{aligned}
MSE &= E[\langle f, f \rangle] - 2E[\langle f, \hat{f} \rangle] + E[\langle \hat{f}, \hat{f} \rangle] \\
&= \sum_i E[n_i^2] \langle \tilde{\psi}_i, \tilde{\psi}_i \rangle \\
&= \epsilon^2 \sum_i \|\tilde{\psi}_i\|^2
\end{aligned} \tag{20}$$

*Theorem.* As in the literature of [6], but we derived in a different way. For frame expansion,

$$\frac{M}{B^2} \leq \sum_k \|\tilde{\psi}_k\|^2 \leq \frac{M}{A^2} \tag{21}$$

*Proof:* Substitute  $f = \psi_k$  into inequalities Eq. 5 and substitute  $f = \tilde{\psi}_k$  into inequalities Eq. 4, we have:

$$\begin{aligned}
\frac{1}{B} \|\psi_k\|^2 &\leq \sum_i |\langle \tilde{\psi}_i, \psi_k \rangle|^2 \leq \frac{1}{A} \|\psi_k\|^2 \\
A \|\tilde{\psi}_k\|^2 &\leq \sum_i |\langle \psi_i, \tilde{\psi}_k \rangle|^2 \leq B \|\tilde{\psi}_k\|^2
\end{aligned}$$

Because for each  $k$ , we have  $\sum_i |\langle \tilde{\psi}_i, \psi_k \rangle|^2$  and  $\sum_i |\langle \psi_i, \tilde{\psi}_k \rangle|^2$  be positive, we have correspondingly:

$$\begin{aligned}
\sum_k \frac{1}{B} \|\psi_k\|^2 &\leq \sum_k \sum_i |\langle \tilde{\psi}_i, \psi_k \rangle|^2 \leq \sum_k \frac{1}{A} \|\psi_k\|^2 \\
\sum_k A \|\tilde{\psi}_k\|^2 &\leq \sum_k \sum_i |\langle \psi_i, \tilde{\psi}_k \rangle|^2 \leq \sum_k B \|\tilde{\psi}_k\|^2
\end{aligned}$$

Since the dimension of the frame basis  $\psi$  is same with the dimension of the dual frame basis  $\tilde{\psi}$ , equals to  $M$ , it is obvious that:

$$\sum_{k=1}^M \sum_{i=1}^M |\langle \tilde{\psi}_i, \psi_k \rangle|^2 = \sum_{k=1}^M \sum_{i=1}^M |\langle \psi_i, \tilde{\psi}_k \rangle|^2 = T$$

Thus  $T$  must satisfy the two inequalities simultaneously

$$\begin{aligned}
\sum_{k=1}^M \frac{1}{B} \|\psi_k\|^2 &\leq T \leq \sum_{k=1}^M \frac{1}{A} \|\psi_k\|^2 \\
\sum_{k=1}^M A \|\tilde{\psi}_k\|^2 &\leq T \leq \sum_{k=1}^M B \|\tilde{\psi}_k\|^2
\end{aligned}$$

This is only possible if

$$\sum_{k=1}^M \frac{1}{A} \|\psi_k\|^2 \geq \sum_{k=1}^M A \|\tilde{\psi}_k\|^2 \quad (22)$$

$$\sum_{k=1}^M B \|\tilde{\psi}_k\|^2 \geq \sum_{k=1}^M \frac{1}{B} \|\psi_k\|^2 \quad (23)$$

Since frame  $\psi_k$  are normalized,  $\|\psi_k\|^2 = 1$ , we have Eq. 22

$$\begin{aligned} \frac{1}{A} \sum_{k=1}^M 1 &= \sum_{k=1}^M \frac{1}{A} \|\psi_k\|^2 \geq \sum_{k=1}^M A \|\tilde{\psi}_k\|^2 \\ \implies \sum_{k=1}^M \|\tilde{\psi}_k\|^2 &\leq \frac{M}{A^2} \end{aligned}$$

Similarly, Eq. 23

$$\begin{aligned} \sum_{k=1}^M B \|\tilde{\psi}_k\|^2 &\geq \sum_{k=1}^M \frac{1}{B} \|\psi_k\|^2 = \frac{1}{B} \sum_{k=1}^M 1 \\ \implies \sum_{k=1}^M \|\tilde{\psi}_k\|^2 &\geq \frac{M}{B^2} \end{aligned}$$

Therefore, the bounds for the  $\sum_k \|\tilde{\psi}_k\|^2$  are established.

$$\frac{M}{B^2} \leq \sum_k \|\tilde{\psi}_k\|^2 \leq \frac{M}{A^2}$$

So that, ■

$$\begin{aligned} MSE &= E[\|f - \hat{f}\|^2] = \epsilon^2 \sum_i \|\tilde{\psi}_i\|^2 \\ \implies \frac{M\epsilon^2}{B^2} &\leq MSE \leq \frac{M\epsilon^2}{A^2} \end{aligned}$$

Note that, since the redundancy of  $r = \frac{M}{N}$  is between the frame bounds  $A$  and  $B$  ( $B > 1$ ), we can guarantee the lower bound of the MSE in the frame expansion case  $\frac{M\epsilon^2}{B^2} = \frac{rN\epsilon^2}{B^2}$  be smaller than the distortion in the orthonormal basis case  $N\epsilon^2$ .

## 4. Tight Frame and Watermarking

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### 4.1 Watermarking

In the spread-spectrum watermarking problem, the signal is transformed using an expansion basis,

$$f = \sum_i \alpha_i \psi_i$$

and the watermark sequence, a white Gaussian noise,  $n_i$ , is added to the coefficients in the transform domain, to form the watermarked image.

$$f' = \sum_i \alpha'_i \psi_i = \sum_i (\alpha_i + \epsilon n_i) \psi_i$$

where  $n_i$  is a zero-mean, unit-variance white Gaussian noise, and  $\epsilon$  is a parameter that controls the watermark strength.

The watermark can be detected assuming the watermark random sequence is known exactly. This is done by performing the forward transform on the watermarked signal, and then running the correlation operation on the coefficients,

$$\rho = \sum_i \hat{\alpha}_i n_i$$

where  $\hat{\alpha}_i$  are the expansion coefficients of the watermarked image  $f'$ .

$$\hat{\alpha}_i = \langle \tilde{\psi}_i, f' \rangle$$

For watermark detection, the correlation is compared to a threshold to decide the presence of the watermark. An optimal threshold can be set to minimize the probability of missing the presence of the watermark error to a given false-detection error according to the Neyman-Pearson criterion [4].

Below, we compare the watermark performance of two procedures: using an *orthonormal basis* and using a *tight-frame* expansion. We ensure these two watermark procedures achieve the same MSE and the same false-alarm error  $P_F$ , and the performance is measured by the minimum missed-detection error,  $P_M$ , as obtained with the Neyman-Pearson procedure.

Three hypothesis cases are possible for the watermark problem [4].

**Case A:** image is not watermarked.

**Case B:** image is watermarked with a random sequence other than the detection watermark, but is using a same watermarking approach.

**Case C:** image is watermarked exactly with the detection watermark.

However, we will not do a *multiple-hypothesis* testing. Instead we will do two *binary hypothesis* testings, on *case A v.s. case C* and *case B v.s. case C* respectively.

## 4.2 Neyman Pearson test[8]

Neyman-Pearson test is also called the *most powerful (MP)* test. The decision rule is obtained by minimizing the *missing error*  $P_M$  subject to the constraint on the *false alarm error*  $P_F$ ,  $P_F = \alpha$ . The objective function is constructed based on the *Lagrange multiplier* method.

$$J = P_M + \lambda(P_F - \alpha)$$

Suppose the decision rule is

$$\begin{array}{c} H_1 \\ \rho \gtrsim T_\rho \\ H_0 \end{array}$$

We derive the Neyman Pearson test on two *binary hypothesis* problems.

#### 4.2.1 No watermarking vs. correct watermarking

According to [4], modeling the correlation  $\rho$  as a normally distributed noise is realistic and fit the central limit theorem.

$$H_0 : \text{Case A} \quad P(\rho|H_0) = \frac{1}{\sqrt{2\pi}\sigma_{\rho A}} e^{-\frac{\rho^2}{2\sigma_{\rho A}^2}}$$

$$H_1 : \text{Case C} \quad P(\rho|H_1) = \frac{1}{\sqrt{2\pi}\sigma_{\rho C}} e^{-\frac{(\rho-\mu_{\rho C})^2}{2\sigma_{\rho C}^2}}$$

False Alarm Error:

$$\begin{aligned} P_F &= P(D_1|H_0) = \int_{T_\rho}^{\infty} P(\rho|H_0) d\rho \\ &= \int_{T_\rho}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{\rho A}} e^{-\frac{\rho^2}{2\sigma_{\rho A}^2}} d\rho \\ &= Q\left(\frac{T_\rho}{\sigma_{\rho A}}\right) \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\sigma_{\rho A}}\right) \end{aligned} \quad (24)$$

Missing Error:

$$\begin{aligned} P_M &= P(D_0|H_1) = \int_{-\infty}^{T_\rho} P(\rho|H_1) d\rho \\ &= \int_{-\infty}^{T_\rho} \frac{1}{\sqrt{2\pi}\sigma_{\rho C}} e^{-\frac{(\rho-\mu_{\rho C})^2}{2\sigma_{\rho C}^2}} d\rho \\ &= G\left(\frac{T_\rho - \mu_{\rho C}}{\sigma_{\rho C}}\right) \\ &= 1 - Q\left(\frac{T_\rho - \mu_{\rho C}}{\sigma_{\rho C}}\right) \\ &= \begin{cases} \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}}\right) & T_\rho \geq \mu_{\rho C} \\ \frac{1}{2} \operatorname{erfc}\left(\frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}}\right) & T_\rho < \mu_{\rho C} \end{cases} \end{aligned} \quad (25)$$

#### 4.2.2 Watermarking Discrimination

$$H_0 : \text{Case B} \quad P(\rho|H_0) = \frac{1}{\sqrt{2\pi}\sigma_{\rho B}} e^{-\frac{\rho^2}{2\sigma_{\rho B}^2}}$$

$$H_1 : \text{Case C} \quad P(\rho|H_1) = \frac{1}{\sqrt{2\pi}\sigma_{\rho C}} e^{-\frac{(\rho-\mu_{\rho C})^2}{2\sigma_{\rho C}^2}}$$

False Alarm Error:

$$P_F = P(D_1|H_0) = \frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\sigma_{\rho B}}\right) \quad (26)$$

Missing Error:

$$P_M = P(D_0|H_1) = \begin{cases} \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}}\right) & T_\rho \geq \mu_{\rho C} \\ \frac{1}{2} \operatorname{erfc}\left(\frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}}\right) & T_\rho < \mu_{\rho C} \end{cases} \quad (27)$$

### 4.3 Correlations

In order to obtain the Neyman-Pearson test for each watermarking procedure, the probability distributions of the watermark correlation under each hypothesis must be obtained first. The correlation distribution is modeled as a Gaussian noise.

#### 4.3.1 Orthonormal basis

**Case A** no watermark

$$\begin{aligned}\rho &= \sum_{i=1}^N \alpha_i n_i \\ \mu_{\rho A} &= E[\rho] = 0\end{aligned}\tag{28}$$

$$\begin{aligned}\sigma_{\rho A}^2 &= E[\rho - \mu_{\rho A}]^2 = E[\rho^2] \\ &= E\left[\sum_i \alpha_i n_i\right]^2 \\ &= E\left[\sum_i \alpha_i^2 n_i^2\right] + E\left[\sum_{i,j,i \neq j} \alpha_i \alpha_j n_i n_j\right] \\ &= \sum_i \alpha_i^2 E[n_i^2] + 0 \\ &= \sum_{i=1}^N \alpha_i^2 = \|f\|^2\end{aligned}\tag{29}$$

**Case B** watermarked with another random sequence

$$\begin{aligned}\rho &= \sum_{i=1}^N (\alpha_i + \epsilon_m m_i) n_i \\ \mu_{\rho B} &= E[\rho] = \sum_i E[\alpha_i n_i] + \sum_i E[\epsilon_m m_i n_i] \\ &= 0 \\ \sigma_{\rho B}^2 &= E[\rho - \mu_{\rho B}]^2 = E[\rho^2] \\ &= E\left[\sum_{i=1}^N (\alpha_i + \epsilon_m m_i) n_i\right]^2 \\ &= E\left[\sum_i (\alpha_i + \epsilon_m m_i)^2 n_i^2\right] + E\left[\sum_{i,j,i \neq j} (\alpha_i + \epsilon_m m_i) n_i \cdot (\alpha_j + \epsilon_m m_j) n_j\right] \\ &= \sum_i E[\alpha_i^2 + \epsilon_m^2 m_i^2 + 2\epsilon_m m_i \alpha_i] E[n_i^2] + 0 \\ &= \sum_{i=1}^N \alpha_i^2 + \sum_{i=1}^N \epsilon_m^2 \\ &= \|f\|^2 + N\epsilon_m^2 = \|f\|^2 + N\epsilon^2\end{aligned}\tag{30}$$

**Case C** watermarked with the correct watermark

$$\begin{aligned}\rho &= \sum_{i=1}^N (\alpha_i + \epsilon n_i) n_i \\ \mu_{\rho C} &= E[\rho] = \epsilon \cdot N\end{aligned}\tag{32}$$

$$\begin{aligned}\sigma_{\rho C}^2 &= E[\rho - \mu_{\rho C}]^2 = E[\rho^2] - \mu_{\rho C}^2 \\ &= E\left[\sum_{i=1}^N (\alpha_i + \epsilon n_i) n_i\right]^2 - \epsilon^2 N^2 \\ &= E\left[\sum_i (\alpha_i n_i + \epsilon n_i^2)^2\right] + E\left[\sum_{i,j,i \neq j} (\alpha_i n_i + \epsilon n_i^2) \cdot (\alpha_j n_j + \epsilon n_j^2)\right] - \epsilon^2 N^2 \\ &= \sum_i E[\alpha_i^2 n_i^2 + \epsilon^2 n_i^4 + 2\epsilon \alpha_i n_i^3] \\ &\quad + E\left[\sum_{i,j,i \neq j} (\alpha_i n_i \alpha_j n_j + \epsilon \alpha_j n_j n_i^2 + \epsilon \alpha_i n_i n_j^2 + \epsilon^2 n_i^2 n_j^2)\right] - \epsilon^2 N^2 \\ &= \sum_i \alpha_i^2 E[n_i^2] + \sum_i \epsilon^2 E[n_i^4] + \sum_i 2\epsilon \alpha_i E[n_i^3] + \sum_{i,j,i \neq j} \epsilon^2 E[n_i^2] E[n_j^2] - \epsilon^2 N^2\end{aligned}$$

Since  $n_i$  is zero mean, unit variance white Gaussian noise, we have:

$$E[n_i^2] = 1 \quad E[n_i^3] = 0 \quad E[n_i^4] = 3$$

$\therefore$

$$\begin{aligned}\sigma_{\rho C}^2 &= \sum_{i=1}^N \alpha_i^2 + 3N\epsilon^2 + \epsilon^2(N^2 - N) - \epsilon^2 N^2 \\ &= \|f\|^2 + 2N\epsilon^2\end{aligned}\tag{33}$$

### 4.3.2 Tight frame

**Case A** no watermark

$$\rho = \sum_{i=1}^{AN} \alpha_i n_i$$
$$\mu_{\rho A} = E[\rho] = 0 \tag{34}$$

$$\sigma_{\rho A}^2 = E[\rho - \mu_{\rho A}]^2 = E[\rho^2]$$
$$= E\left[\sum_{i=1}^{AN} \alpha_i n_i\right]^2$$
$$= \sum_i \alpha_i^2 E[n_i^2]$$
$$= \sum_{i=1}^{AN} \alpha_i^2 = A\|f\|^2 \tag{35}$$

**Case B** watermarked with another random sequence

$$\begin{aligned}
\rho &= \sum_{i=1}^{AN} \hat{\alpha}_i n_i \\
&= \sum_i \langle \psi_i, f' \rangle n_i = \sum_i \left\langle \psi_i, \frac{1}{A} \sum_j (\alpha_j + \epsilon_m m_j) \psi_j \right\rangle n_i \\
&= \frac{1}{A} \sum_i \left[ \sum_j (\alpha_j + \epsilon_m m_j) \langle \psi_i, \psi_j \rangle \right] n_i \\
&= \frac{1}{A} \sum_i \sum_j (\alpha_j + \epsilon_m m_j) n_i \langle \psi_i, \psi_j \rangle \\
\mu_{\rho B} &= E[\rho] = 0 \\
\sigma_{\rho B}^2 &= E[\rho - \mu_{\rho B}]^2 = E[\rho^2] \\
&= E\left[ \frac{1}{A} \sum_i \sum_j (\alpha_j + \epsilon_m m_j) n_i \langle \psi_i, \psi_j \rangle \right]^2 \\
&= \frac{1}{A^2} \left[ E\left[ \sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle \right]^2 + E\left[ \sum_i \sum_j \epsilon_m m_j n_i \langle \psi_i, \psi_j \rangle \right]^2 \right. \\
&\quad \left. + 2E\left[ \sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle \cdot \sum_k \sum_l \epsilon_m m_l n_k \langle \psi_k, \psi_l \rangle \right] \right] \\
&= \frac{1}{A^2} E\left[ \sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle \right]^2 + \frac{1}{A^2} E\left[ \sum_i \sum_j \epsilon_m m_j n_i \langle \psi_i, \psi_j \rangle \right]^2 \\
&= \frac{1}{A^2} \sum_i E[n_i^2] \left[ \sum_j \alpha_j \langle \psi_i, \psi_j \rangle \right]^2 + \frac{1}{A^2} \sum_i \sum_j \epsilon_m^2 E[m_j^2 n_i^2] \langle \psi_i, \psi_j \rangle^2 \\
&= \frac{1}{A^2} \sum_{i=1}^{AN} \left[ \sum_j \alpha_j \langle \psi_i, \psi_j \rangle \right]^2 + \frac{\epsilon_m^2}{A^2} \sum_{i=1}^{AN} \sum_{j=1}^{AN} \langle \psi_i, \psi_j \rangle^2
\end{aligned} \tag{36}$$

Since for the tight frame, we have:

$$f = \frac{1}{A} \sum_j \langle \psi_j, f \rangle \psi_j = \frac{1}{A} \sum_j \alpha_j \psi_j$$

So that,

$$\begin{aligned}
\sum_{i=1}^{AN} \left[ \sum_j \alpha_j \langle \psi_i, \psi_j \rangle \right]^2 &= \sum_{i=1}^{AN} \langle \psi_i, Af \rangle^2 \\
&= \sum_{i=1}^{AN} A^2 \langle \psi_i, f \rangle^2 \\
&= A^2 \sum_{i=1}^{AN} \alpha_i^2
\end{aligned} \tag{37}$$

And also because tight frame has the property of:

$$\sum_i |\langle \psi_i, f \rangle|^2 = A \|f\|^2$$

Then we have the second item of the expression:

$$\begin{aligned}\sum_{i=1}^{AN} \sum_{j=1}^{AN} \langle \psi_i, \psi_j \rangle^2 &= \sum_{i=1}^{AN} A \|\psi_i\|^2 \\ &= \sum_{i=1}^{AN} A = A^2 N\end{aligned}$$

∴

$$\begin{aligned}\sigma_{\rho B}^2 &= \frac{1}{A^2} \cdot A^2 \sum_{i=1}^{AN} \alpha_i^2 + \frac{\epsilon_m^2}{A^2} \cdot A^2 N \\ &= \sum_{i=1}^{AN} \alpha_i^2 + \epsilon_m^2 N \\ &= A \|f\|^2 + \epsilon_m^2 N = A \|f\|^2 + \epsilon^2 N\end{aligned}\tag{38}$$

**Case C** watermarked with the correct watermark

$$\begin{aligned}\rho &= \frac{1}{A} \sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle + \frac{1}{A} \sum_i \sum_j \epsilon n_j n_i \langle \psi_i, \psi_j \rangle \\ \mu_{\rho C} &= E[\rho] = \epsilon \cdot N \\ \sigma_{\rho C}^2 &= E[\rho - \mu_{\rho C}]^2 = E[\rho^2] - \mu_{\rho C}^2 \\ &= \frac{1}{A^2} \left[ E\left[\sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle\right]^2 + E\left[\sum_i \sum_j \epsilon n_j n_i \langle \psi_i, \psi_j \rangle\right]^2 \right. \\ &\quad \left. + 2E\left[\sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle \cdot \sum_k \sum_l \epsilon n_l n_k \langle \psi_k, \psi_l \rangle\right] \right] - \epsilon^2 N^2 \\ &= \frac{1}{A^2} \left[ \sum_i E[n_i^2] \left[\sum_j \alpha_j \langle \psi_i, \psi_j \rangle\right]^2 \right. \\ &\quad \left. + E\left[\sum_i \sum_j \epsilon n_j n_i \langle \psi_i, \psi_j \rangle \cdot \sum_k \sum_l \epsilon n_l n_k \langle \psi_k, \psi_l \rangle\right] \right. \\ &\quad \left. + 2E\left[\sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle \cdot \sum_k \sum_l \epsilon n_l n_k \langle \psi_k, \psi_l \rangle\right] \right] \\ &\quad - \epsilon^2 N^2\end{aligned}\tag{39}$$

For each of the item in the expression,

(1) From the Eq. 37, we have

$$\begin{aligned}\sum_i E[n_i^2] \left[\sum_j \alpha_j \langle \psi_i, \psi_j \rangle\right]^2 &= \sum_{i=1}^{AN} \left[\sum_j \alpha_j \langle \psi_i, \psi_j \rangle\right]^2 \\ &= A^2 \sum_{i=1}^{AN} \alpha_i^2\end{aligned}$$

(2) The second item in the expression

$$\begin{aligned}
& E\left[\sum_i \sum_j \epsilon n_j n_i \langle \psi_i, \psi_j \rangle \cdot \sum_k \sum_l \epsilon n_l n_k \langle \psi_k, \psi_l \rangle\right] \\
&= E\left[\sum_i \epsilon^2 n_i^4 \langle \psi_k, \psi_l \rangle^2\right] \quad \leftarrow i = j = k = l \\
&+ E\left[\sum_i \sum_k \epsilon^2 n_i^2 \langle \psi_i, \psi_i \rangle \cdot n_k^2 \langle \psi_k, \psi_k \rangle\right] \quad \leftarrow i = j, k = l, i \neq k \\
&+ 2E\left[\sum_i \sum_j \epsilon^2 n_j^2 n_i^2 \langle \psi_i, \psi_j \rangle^2\right] \quad \leftarrow \begin{array}{l} i = k, j = l, i \neq j \\ \text{or } i = l, j = k, i \neq j \end{array} \\
&+ 0 \quad \leftarrow \text{else} \\
&= \epsilon^2 \sum_i E[n_i^4] + \epsilon^2 \sum_{i,k,i \neq k} E[n_i^2] E[n_k^2] + 2\epsilon^2 \sum_{i,j,i \neq j} E[n_i^2] E[n_j^2] \langle \psi_i, \psi_j \rangle^2 \\
&= 3AN\epsilon^2 + \epsilon^2(A^2N^2 - AN) + 2\epsilon^2 \left[ \sum_{i=1}^{AN} \sum_{j=1}^{AN} \langle \psi_i, \psi_j \rangle^2 - \sum_{i=1}^{AN} \langle \psi_i, \psi_i \rangle^2 \right] \\
&= 3AN\epsilon^2 + \epsilon^2(A^2N^2 - AN) + 2\epsilon^2(A^2N - AN) \\
&= A^2\epsilon^2N^2 + 2\epsilon^2A^2N \\
&= \epsilon^2A^2(N^2 + 2N)
\end{aligned}$$

(3) The third item in the expression

$$\begin{aligned}
& E\left[\sum_i \sum_j \alpha_j n_i \langle \psi_i, \psi_j \rangle \cdot \sum_k \sum_l \epsilon n_l n_k \langle \psi_k, \psi_l \rangle\right] \\
&= E\left[\sum_i \sum_j \epsilon \alpha_j n_i^3 \langle \psi_i, \psi_j \rangle \langle \psi_i, \psi_i \rangle\right] \quad \leftarrow i = k = l \\
&+ 0 \quad \leftarrow \text{else} \\
&= 0
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\sigma_{\rho_C}^2 &= \frac{1}{A^2} \left[ A^2 \sum_{i=1}^{AN} \alpha_i^2 + \epsilon^2 A^2 (N^2 + 2N) \right] - \epsilon^2 N^2 \\
&= \sum_{i=1}^{AN} \alpha_i^2 + \epsilon^2 (N^2 + 2N) - \epsilon^2 N^2 \\
&= A \|f\|^2 + 2\epsilon^2 N
\end{aligned} \tag{40}$$

#### 4.4 Performance comparison

We know the MSE incurred by the watermarking process:

$$\text{Orthonormal Basis: } MSE = E[\|f - f'\|^2] = N\epsilon^2 = D$$

$$\text{Tight Frame: } MSE = E[\|f - f'\|^2] = \frac{N\epsilon^2}{A}$$

$$\text{Frame Basis: } \frac{M\epsilon^2}{B^2} \leq MSE = E[\|f - f'\|^2] \leq \frac{M\epsilon^2}{A^2}$$

To achieve same MSE's for the three expansions, the watermarking strength  $\epsilon$  must be adjusted as:

$$\text{Orthonormal Basis: } \epsilon = \sqrt{\frac{D}{N}} \quad (41)$$

$$\text{Tight Frame: } \epsilon = \sqrt{\frac{AD}{N}} \quad (42)$$

$$\text{Frame Basis: } \sqrt{\frac{D}{M}}A \leq \epsilon \leq \sqrt{\frac{D}{M}}B \quad (43)$$

The statistical parameters of the Gaussian modeled correlations are listed in Table 4.4.

**Table 1:** Mean and standard deviation of the watermark correlation.

	Orthonormal Basis	Tight Frame
Case A	$\mu_{\rho A} = 0$ $\sigma_{\rho A}^2 = \ f\ ^2$	$\mu'_{\rho A} = 0$ $\sigma'_{\rho A}{}^2 = A\ f\ ^2$
Case B	$\mu_{\rho B} = 0$ $\sigma_{\rho B}^2 = \ f\ ^2 + N\epsilon^2 = \ f\ ^2 + D$	$\mu'_{\rho B} = 0$ $\sigma'_{\rho B}{}^2 = A\ f\ ^2 + N\epsilon'^2 = A\ f\ ^2 + AD$
Case C	$\mu_{\rho C} = \epsilon \cdot N = \sqrt{ND}$ $\sigma_{\rho C}^2 = \ f\ ^2 + 2N\epsilon^2 = \ f\ ^2 + 2D$	$\mu'_{\rho C} = \epsilon' \cdot N = \sqrt{AND}$ $\sigma'_{\rho C}{}^2 = A\ f\ ^2 + 2N\epsilon'^2 = A\ f\ ^2 + 2AD$

#### 4.4.1 No watermarking vs. correct watermarking

$H_0$  : Case A

$H_1$  : Case C

For Orthonormal Basis:

$$\begin{aligned} P_F &= \frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\sigma_{\rho A}}\right) \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\|f\|^2}\right) \end{aligned}$$

For Tight Frame:

$$\begin{aligned} P'_F &= \frac{1}{2} \operatorname{erfc}\left(\frac{T'_\rho}{\sqrt{2}\sigma'_{\rho A}}\right) \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{T'_\rho}{\sqrt{2A}\|f\|^2}\right) \end{aligned}$$

In order for the false alarm error be the same,  $P_F = P'_F$ , we need

$$\frac{T_\rho}{\sqrt{2}\|f\|^2} = \frac{T'_\rho}{\sqrt{2A}\|f\|^2}$$

So that  $T'_\rho = \sqrt{A}T_\rho$

Since the missing error is:

$$P_M = \begin{cases} \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}}\right) & T_\rho \geq \mu_{\rho C} \\ \frac{1}{2} \operatorname{erfc}\left(\frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}}\right) & T_\rho < \mu_{\rho C} \end{cases}$$

(1) if  $T_\rho \geq \mu_{\rho C}$ , then  $T'_\rho = \sqrt{A}T_\rho \geq \sqrt{A}\mu_{\rho C} = \mu'_{\rho C}$

We have

$$P_M = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}}\right)$$

$$P'_M = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T'_\rho - \mu'_{\rho C}}{\sqrt{2}\sigma'_{\rho C}}\right)$$

Since

$$\begin{aligned} \frac{T'_\rho - \mu'_{\rho C}}{\sqrt{2}\sigma'_{\rho C}} &= \frac{\sqrt{A}T_\rho - \sqrt{A}\mu_{\rho C}}{\sqrt{2}\sqrt{A}\|f\|^2 + 2AN\sigma^2} \\ &= \frac{\sqrt{A}T_\rho - \sqrt{A}\mu_{\rho C}}{\sqrt{2}\sqrt{A}\sigma_{\rho C}} \\ &= \frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}} \end{aligned}$$

Therefore, we have  $P_M = P'_M$ .

(2) Similarly, if  $T_\rho < \mu_{\rho C}$ , we have  $T'_\rho < \mu'_{\rho C}$ , and

$$P_M = \frac{1}{2} \operatorname{erfc}\left(\frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}}\right)$$

$$P'_M = \frac{1}{2} \operatorname{erfc}\left(\frac{\mu'_{\rho C} - T'_\rho}{\sqrt{2}\sigma'_{\rho C}}\right)$$

Since,

$$\frac{\mu'_{\rho C} - T'_\rho}{\sqrt{2}\sigma'_{\rho C}} = \frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}}$$

we have  $P_M = P'_M$  also.

That is, the probability of missed detection error is the same regardless of whether an orthonormal or tight frame is used.

#### 4.4.2 Watermarking Discrimination

$H_0$  : Case B

$H_1$  : Case C

For Orthonormal Basis:

$$P_F = \frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\sigma_{\rho B}}\right)$$

$$= \frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\sqrt{\|f\|^2 + D}}\right)$$

For Tight Frame:

$$\begin{aligned} P'_F &= \frac{1}{2} \operatorname{erfc}\left(\frac{T'_\rho}{\sqrt{2}\sigma'_{\rho B}}\right) \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{T'_\rho}{\sqrt{2}\sqrt{A\|f\|^2 + AD}}\right) \end{aligned}$$

In order for  $P_F = P'_F$ , we need

$$\frac{1}{2} \operatorname{erfc}\left(\frac{T_\rho}{\sqrt{2}\sqrt{\|f\|^2 + D}}\right) = \frac{1}{2} \operatorname{erfc}\left(\frac{T'_\rho}{\sqrt{2}\sqrt{A\|f\|^2 + AD}}\right)$$

So that

$$T'_\rho = \sqrt{A}T_\rho$$

(1) if  $T_\rho \geq \mu_{\rho C}$ , then  $T'_\rho = \sqrt{A}T_\rho \geq \sqrt{A}\mu_{\rho C} = \mu'_{\rho C}$

we have

$$\begin{aligned} P_M &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}}\right) \\ P'_M &= \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{T'_\rho - \mu'_{\rho C}}{\sqrt{2}\sigma'_{\rho C}}\right) \end{aligned}$$

$$\begin{aligned} \frac{T'_\rho - \mu'_{\rho C}}{\sqrt{2}\sigma'_{\rho C}} &= \frac{\sqrt{A}T_\rho - \sqrt{A}\mu_{\rho C}}{\sqrt{2}\sqrt{A}\sigma_{\rho C}} \\ &= \frac{T_\rho - \mu_{\rho C}}{\sqrt{2}\sigma_{\rho C}} \end{aligned}$$

Therefore, we have  $P_M = P'_M$ .

(2) if  $T_\rho < \mu_{\rho C}$ , we have  $T'_\rho < \mu'_{\rho C}$ , and

$$\begin{aligned} P_M &= \frac{1}{2} \operatorname{erfc}\left(\frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}}\right) \\ P'_M &= \frac{1}{2} \operatorname{erfc}\left(\frac{\mu'_{\rho C} - T'_\rho}{\sqrt{2}\sigma'_{\rho C}}\right) \end{aligned}$$

$$\begin{aligned} \frac{\mu'_{\rho C} - T'_\rho}{\sqrt{2}\sigma'_{\rho C}} &= \frac{\sqrt{A}\mu_{\rho C} - \sqrt{A}T_\rho}{\sqrt{2}\sqrt{A}\sigma_{\rho C}} \\ &= \frac{\mu_{\rho C} - T_\rho}{\sqrt{2}\sigma_{\rho C}} \end{aligned}$$

So that we will have  $P_M = P'_M$  also.

Again, the performance is the same.

## 5. Conclusions

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After the above theoretical analysis, we can draw the following conclusions:

1. RDWT expansion is a *frame* expansion, and one scale RDWT is a *tight frame* with the redundancy  $A = 2$ .
2. *Frame* expansion is more robust than the *orthonormal basis* expansion when adding white noise onto the transform domain; i.e., less distortion is obtained when water-marking with a fixed watermark energy.
3. However, *tight frame* doesn't show obvious performance advantages over the *orthonormal basis* when considering the spread-spectrum watermarking problems, as the additional robustness of the overcomplete expansion does not aid watermark detection by correlation operators.

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